Access to Science, Engineering and Agriculture: Mathematics 1 MATH00030 Chapter 3 Solutions

1. (a) $x^2 + 4x - 5 = [x^2 + 4x] - 5 = [(x+2)^2 - 4] - 5 = (x+2)^2 - 9.$

   (b) $x^2 - 8x - 20 = [x^2 - 8x] - 20 = [(x-4)^2 - 16] - 20 = (x-4)^2 - 36.$

   (c) $x^2 + 5x - 6 = [x^2 + 5x] - 6 = \left[\left(x + \frac{5}{2}\right)^2 - \frac{25}{4}\right] - 6 = \left(x + \frac{5}{2}\right)^2 - \frac{49}{4}.$

   (d) $x^2 - 7x + 2 = [x^2 - 7x] + 2 = \left[\left(x - \frac{7}{2}\right)^2 - \frac{49}{4}\right] + 2 = \left(x - \frac{7}{2}\right)^2 - \frac{41}{4}.$

   (e) $2x^2 + 3x = 2\left\{x^2 + \frac{3}{2}x\right\} = 2\left\{\left(x + \frac{3}{4}\right)^2 - \frac{9}{16}\right\} = 2\left(x + \frac{3}{4}\right)^2 - \frac{9}{8}.$

   (f)

$$\begin{aligned}
-3x^2 + 5x - 1 &= -3\left\{x^2 - \frac{5}{3}x + \frac{1}{3}\right\} \\
&= -3\left\{\left[x^2 - \frac{5}{3}x\right] + \frac{1}{3}\right\} \\
&= -3\left\{\left[\left(x - \frac{5}{6}\right)^2 - \frac{25}{36}\right] + \frac{1}{3}\right\} \\
&= -3\left\{\left(x - \frac{5}{6}\right)^2 - \frac{13}{36}\right\} \\
&= -3\left(x - \frac{5}{6}\right)^2 + \frac{13}{12}.
\end{aligned}$$

   (g)

$$\begin{aligned}
\frac{1}{3}x^2 - \frac{1}{4}x - \frac{2}{3} &= \frac{1}{3}\left\{x^2 - \frac{3}{4}x - 2\right\} \\
&= \frac{1}{3}\left\{\left[x^2 - \frac{3}{4}x\right] - 2\right\} \\
&= \frac{1}{3}\left\{\left[\left(x - \frac{3}{8}\right)^2 - \frac{9}{64}\right] - 2\right\} \\
&= \frac{1}{3}\left\{\left(x - \frac{3}{8}\right)^2 - \frac{137}{64}\right\} \\
&= \frac{1}{3}\left(x - \frac{3}{8}\right)^2 - \frac{137}{192}.
\end{aligned}$$

(h)

$$\begin{aligned}
-\frac{3}{4}x^2+2x-\frac{1}{5} &= -\frac{3}{4}\left\{x^2-\frac{8}{3}x+\frac{4}{15}\right\} \\
&= -\frac{3}{4}\left\{\left[x^2-\frac{8}{3}x\right]+\frac{4}{15}\right\} \\
&= -\frac{3}{4}\left\{\left[\left(x-\frac{4}{3}\right)^2-\frac{16}{9}\right]+\frac{4}{15}\right\} \\
&= -\frac{3}{4}\left\{\left(x-\frac{4}{3}\right)^2-\frac{68}{45}\right\} \\
&= -\frac{3}{4}\left(x-\frac{4}{3}\right)^2+\frac{17}{15}.
\end{aligned}$$

2. In each case we will start by rewriting the equation using the completed square form found in Question 1.

(a)

$$\begin{aligned}
x^2+4x-5=0 &\Rightarrow (x+2)^2-9=0 \\
&\Rightarrow (x+2)^2=9 \\
&\Rightarrow x+2=\pm 3 \\
&\Rightarrow x=-5 \text{ or } x=1.
\end{aligned}$$

(b)

$$\begin{aligned}
x^2-8x-20=0 &\Rightarrow (x-4)^2-36=0 \\
&\Rightarrow (x-4)^2=36 \\
&\Rightarrow x-4=\pm 6 \\
&\Rightarrow x=-2 \text{ or } x=10.
\end{aligned}$$

(c)

$$\begin{aligned}
x^2+5x-6=0 &\Rightarrow \left(x+\frac{5}{2}\right)^2-\frac{49}{4}=0 \\
&\Rightarrow \left(x+\frac{5}{2}\right)^2=\frac{49}{4} \\
&\Rightarrow x+\frac{5}{2}=\pm\frac{7}{2} \\
&\Rightarrow x=-6 \text{ or } x=1.
\end{aligned}$$

(d)

$$\begin{aligned} x^2 - 7x + 2 = 0 &\Rightarrow \left(x - \frac{7}{2}\right)^2 - \frac{41}{4} = 0 \\ &\Rightarrow \left(x - \frac{7}{2}\right)^2 = \frac{41}{4} \\ &\Rightarrow x - \frac{7}{2} = \pm\frac{\sqrt{41}}{2} \\ &\Rightarrow x = \frac{7 - \sqrt{41}}{2} \text{ or } x = \frac{7 + \sqrt{41}}{2}. \end{aligned}$$

(e)

$$\begin{aligned} 2x^2 + 3x = 0 &\Rightarrow 2\left(x + \frac{3}{4}\right)^2 - \frac{9}{8} = 0 \\ &\Rightarrow 2\left(x + \frac{3}{4}\right)^2 = \frac{9}{8} \\ &\Rightarrow \left(x + \frac{3}{4}\right)^2 = \frac{9}{16} \\ &\Rightarrow x + \frac{3}{4} = \pm\frac{3}{4} \\ &\Rightarrow x = -\frac{3}{2} \text{ or } x = 0. \end{aligned}$$

(f)

$$\begin{aligned} -3x^2 + 5x - 1 = 0 &\Rightarrow -3\left(x - \frac{5}{6}\right)^2 + \frac{13}{12} = 0 \\ &\Rightarrow -3\left(x - \frac{5}{6}\right)^2 = -\frac{13}{12} \\ &\Rightarrow \left(x - \frac{5}{6}\right)^2 = \frac{13}{36} \\ &\Rightarrow x - \frac{5}{6} = \pm\frac{\sqrt{13}}{6} \\ &\Rightarrow x = \frac{5 - \sqrt{13}}{6} \text{ or } x = \frac{5 + \sqrt{13}}{6}. \end{aligned}$$

(g)

$$\begin{aligned}\frac{1}{3}x^2-\frac{1}{4}x-\frac{2}{3}=0 &\Rightarrow \frac{1}{3}\left(x-\frac{3}{8}\right)^2-\frac{137}{192}=0\\ &\Rightarrow \frac{1}{3}\left(x-\frac{3}{8}\right)^2=\frac{137}{192}\\ &\Rightarrow \left(x-\frac{3}{8}\right)^2=\frac{137}{64}\\ &\Rightarrow x-\frac{3}{8}=\pm\frac{\sqrt{137}}{8}\\ &\Rightarrow x=\frac{3-\sqrt{137}}{8} \text{ or } x=\frac{3+\sqrt{137}}{8}.\end{aligned}$$

(h)

$$\begin{aligned}-\frac{3}{4}x^2+2x-\frac{1}{5}=0 &\Rightarrow -\frac{3}{4}\left(x-\frac{4}{3}\right)^2+\frac{17}{15}=0\\ &\Rightarrow -\frac{3}{4}\left(x-\frac{4}{3}\right)^2=-\frac{17}{15}\\ &\Rightarrow \left(x-\frac{4}{3}\right)^2=\frac{68}{45}\\ &\Rightarrow x-\frac{4}{3}=\pm\frac{\sqrt{68}}{\sqrt{45}}\\ &\Rightarrow x=\frac{4}{3}\pm\frac{\sqrt{68}}{\sqrt{45}}\\ &\Rightarrow x=\frac{20}{15}\pm\frac{\sqrt{340}}{15}\\ &\Rightarrow x=\frac{20}{15}\pm\frac{2\sqrt{85}}{15}\\ &\Rightarrow x=\frac{20-2\sqrt{85}}{15} \text{ or } x=\frac{20+2\sqrt{85}}{15}.\end{aligned}$$

Note that in this case the answer $x=\frac{4}{3}\pm\frac{\sqrt{68}}{\sqrt{45}}$ would be fine.
The 'tidying up' is a bit tricky!

3. (a) In this case $a=1$, $b=1$ and $c=-1$.
Hence the solutions of the equation are

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-1\pm\sqrt{1^2-4(1)(-1)}}{2(1)}=\frac{-1\pm\sqrt{1+4}}{2}=\frac{-1\pm\sqrt{5}}{2}.$$

(b) In this case $a = 1$, $b = -1$ and $c = 1$.
Hence the solutions of the equation are

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)} \\ &= \frac{1 \pm \sqrt{1-4}}{2} \\ &= \frac{1 \pm \sqrt{-3}}{2} \\ &= \frac{1 \pm \sqrt{3}i}{2}. \end{aligned}$$

(c) In this case $a = -4$, $b = -1$ and $c = 3$.
Hence the solutions of the equation are

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(-4)(3)}}{2(-4)} \\ &= \frac{1 \pm \sqrt{1+48}}{-8} \\ &= \frac{1 \pm \sqrt{49}}{-8} \\ &= \frac{1 \pm 7}{-8} \\ &= \frac{3}{4} \text{ or } -1. \end{aligned}$$

(d) In this case $a = 1$, $b = 0$ and $c = 1$.
Hence the solutions of the equation are

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-0 \pm \sqrt{0^2 - 4(1)(1)}}{2(1)} \\ &= \frac{\pm\sqrt{-4}}{2} \\ &= \frac{\pm\sqrt{4}i}{2} \\ &= \frac{\pm 2i}{2} \\ &= \pm i. \end{aligned}$$

(e) In this case $a = -3$, $b = 4$ and $c = 0$.
Hence the solutions of the equation are

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-4 \pm \sqrt{4^2 - 4(-3)(0)}}{2(-3)} \\
&= \frac{-4 \pm \sqrt{16}}{-6} \\
&= \frac{-4 \pm 4}{-6} \\
&= \frac{4}{3} \text{ or } 0.
\end{aligned}
$$

(f) In this case $a = \frac{1}{5}$, $b = -\frac{1}{4}$ and $c = \frac{1}{3}$.
Hence the solutions of the equation are

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-\left(-\frac{1}{4}\right) \pm \sqrt{\left(-\frac{1}{4}\right)^2 - 4\left(\frac{1}{5}\right)\left(\frac{1}{3}\right)}}{2\left(\frac{1}{5}\right)} \\
&= \frac{\frac{1}{4} \pm \sqrt{\frac{1}{16} - \frac{4}{15}}}{2/5} \\
&= \frac{\frac{1}{4} \pm \sqrt{-\frac{49}{240}}}{2/5} \\
&= \frac{\frac{1}{4} \pm \sqrt{\frac{49}{240}}i}{2/5} \\
&= \frac{\frac{1}{4} \pm \frac{7}{\sqrt{240}}i}{2/5} \\
&= \frac{\frac{1}{4} \pm \frac{7\sqrt{15}}{60}i}{2/5} \\
&= \frac{5}{8} + \frac{7\sqrt{15}}{24}i \\
&= \frac{15 \pm 7\sqrt{15}i}{24}.
\end{aligned}
$$

(g) In this case $a = 1$, $b = 4$ and $c = 4$.
Hence the solution of the equation is

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-4 \pm \sqrt{4^2 - 4(1)(4)}}{2(1)} \\
&= \frac{-4 \pm \sqrt{0}}{2} \\
&= \frac{-4}{2} \\
&= -2.
\end{aligned}
$$

4. (a) From Question 3a we know that the graph cuts the x-axis when $x = \dfrac{-1 - \sqrt{5}}{2}$ and when $x = \dfrac{-1 + \sqrt{5}}{2}$.
Next, when $x = 0$, $y = -1$, so the graph cuts the y-axis when $y = -1$.
We also know the graph is U-shaped since $a > 0$.
Finally, the turning point is given by

$$
\left(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a} \right) = \left(-\frac{1}{2(1)}, -\frac{1^2 - 4(1)(-1)}{4(1)} \right) = \left(-\frac{1}{2}, -\frac{5}{4} \right).
$$

We now have all the information we need and I have sketched the graph in Figure 1a below.

(b) From Question 3b we know that the graph does not cut the x-axis since the solutions of the equation $y = x^2 - x + 1 = 0$ are complex.
Next, when $x = 0$, $y = 1$, so the graph cuts the y-axis when $y = 1$.
We also know the graph is U-shaped since $a > 0$.
Finally, the turning point is given by

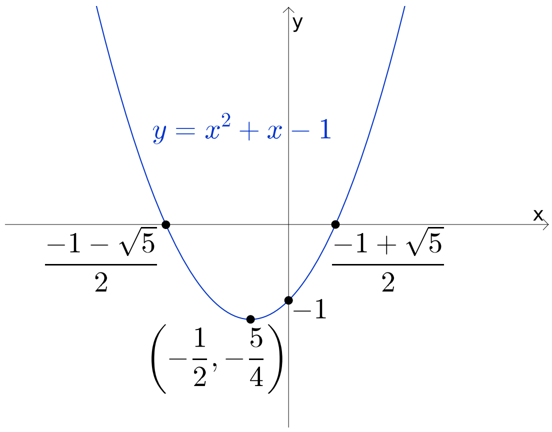
$$
\left(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a} \right) = \left(-\frac{-1}{2(1)}, -\frac{(-1)^2 - 4(1)(1)}{4(1)} \right) = \left(\frac{1}{2}, \frac{3}{4} \right).
$$

We now have all the information we need and I have sketched the graph in Figure 1b.

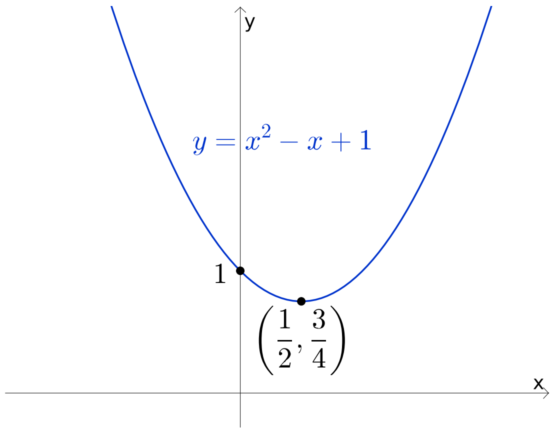
(c) From Question 3c we know that the graph cuts the x-axis when $x = \dfrac{3}{4}$ and $x = -1$.
Next, when $x = 0$, $y = 3$, so the graph cuts the y-axis when $y = 3$.
We also know the graph is shaped like an upside down U since $a < 0$.
Finally, the turning point is given by

$$
\left(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a} \right) = \left(-\frac{-1}{2(-4)}, -\frac{(-1)^2 - 4(-4)(3)}{4(-4)} \right) = \left(-\frac{1}{8}, \frac{49}{16} \right).
$$

We now have all the information we need and I have sketched the graph in Figure 2a below.



(a) A sketch of the graph of the function $y = x^2 + x - 1$.



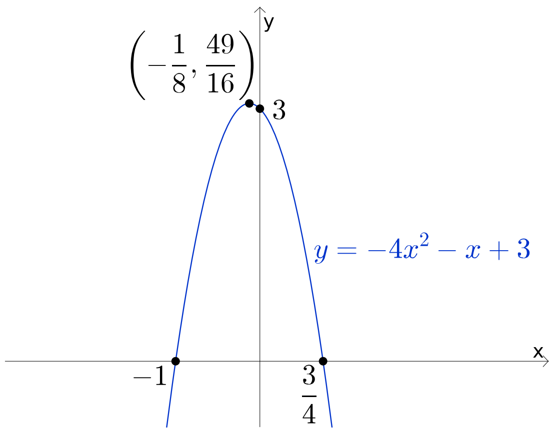
(b) A sketch of the graph of the function $y = x^2 - x + 1$.

Figure 1

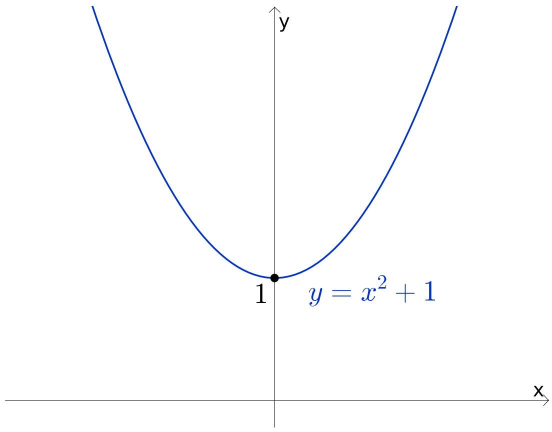
(d) From Question 3d we know that the graph does not cut the x-axis since the solutions of the equation $y = x^2 + 1 = 0$ are complex.
Next, when $x = 0$, $y = 1$, so the graph cuts the y-axis when $y = 1$.
We also know the graph is U-shaped since $a > 0$.
Finally, the turning point is given by

$$\left(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a}\right) = \left(-\frac{0}{2(1)}, -\frac{0^2 - 4(1)(1)}{4(1)}\right) = (0, 1)\,.$$

We now have all the information we need and I have sketched the graph in Figure 2b.



(a) A sketch of the graph of the function $y = -4x^2 - x + 3$.



(b) A sketch of the graph of the function $y = x^2 + 1$.

Figure 2

(e) From Question 3e we know that the graph cuts the x-axis when $x = \frac{4}{3}$ and $x = 0$.
Next, when $x = 0$, $y = 0$, so the graph cuts the y-axis when $y = 0$ (note that we already know this since one of the x-intercepts occurs when $x = 0$).

We also know the graph is shaped like an upside down U since $a < 0$.
Finally, the turning point is given by

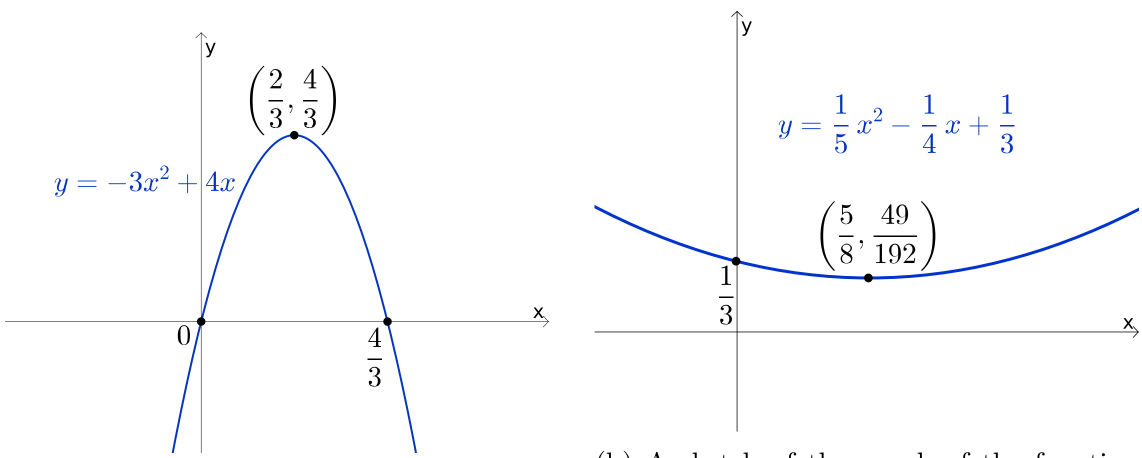
$$\left(-\frac{b}{2a}, -\frac{b^2-4ac}{4a}\right) = \left(-\frac{4}{2(-3)}, -\frac{4^2-4(-3)(0)}{4(-3)}\right) = \left(\frac{2}{3}, \frac{4}{3}\right).$$

We now have all the information we need and I have sketched the graph in Figure 3a below.

(f) From Question 3f we know that the graph does not cut the x-axis since the solutions of the equation $y = \frac{1}{5}x^2 - \frac{1}{4}x + \frac{1}{3} = 0$ are complex.
Next, when $x = 0$, $y = \frac{1}{3}$, so the graph cuts the y-axis when $y = \frac{1}{3}$.
We also know the graph is U-shaped since $a > 0$.
Finally, the turning point is given by

$$\left(-\frac{b}{2a}, -\frac{b^2-4ac}{4a}\right) = \left(-\frac{-\frac{1}{4}}{2\left(\frac{1}{5}\right)}, -\frac{\left(-\frac{1}{4}\right)^2 - 4\left(\frac{1}{5}\right)\left(\frac{1}{3}\right)}{4\left(\frac{1}{5}\right)}\right) = \left(\frac{5}{8}, \frac{49}{192}\right).$$

We now have all the information we need and I have sketched the graph in Figure 3b.



(a) A sketch of the graph of the function $y = -3x^2 + 4x$.

(b) A sketch of the graph of the function $y = \frac{1}{5}x^2 - \frac{1}{4}x + \frac{1}{3}$.

Figure 3

(g) From Question 3g we know that the graph touches the x-axis when $x = -2$.
Next, when $x = 0$, $y = 4$, so the graph cuts the y-axis when $y = 4$.
We also know the graph is U-shaped since $a > 0$.
Finally, the turning point is given by

$$\left(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a}\right) = \left(-\frac{4}{2(1)}, -\frac{4^2 - 4(1)(4)}{4(1)}\right) = (-2, 0)\,.$$

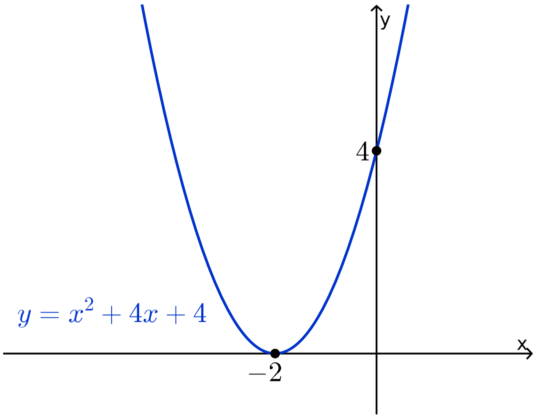


Figure 4: A sketch of the graph of the function $y = x^2 + 4x + 4$.

We now have all the information we need and I have sketched the graph in Figure 4.
Note that if the graph just touches the x-axis then the turning point will always be at this point.

5. In each of these questions we will use the solutions to Question 3. If there is a factorization then we will change the signs of the solutions and also multiply by the coefficient of x^2 if it is not 1.

(a) $x^2 + x - 1 = \left(x - \frac{-1-\sqrt{5}}{2}\right)\left(x - \frac{-1+\sqrt{5}}{2}\right)$.
You could also write this as $\left(x + \frac{1+\sqrt{5}}{2}\right)\left(x + \frac{1-\sqrt{5}}{2}\right)$ but this is not necessary.

(b) The solutions in Question 3b are complex, so we say in this course that $x^2 - x + 1$ can't be factorized. Note that in certain courses, complex factorizations may be allowed.

(c) $-4x^2 - x + 3 = -4\left(x - \frac{3}{4}\right)(x - (-1)) = (-4x + 3)(x + 1)$.
Here I multiplied the -4 by the first bracket since it got rid of the fraction.

(d) $x^2 + 1$ can't be factorized since the solutions in Question 3d are complex.

(e) $-3x^2 + 4x = -3(x - 0)\left(x - \frac{4}{3}\right) = x(-3x + 4)$.
Here I multiplied the second term by -3 to get rid of the fraction.

(f) $\frac{1}{5}x^2 - \frac{1}{4}x + \frac{1}{3}$ can't be factorized since the solutions in Question 3f are complex.

(g) $x^2 + 4x + 4 = (x - (-2))(x - (-2)) = (x + 2)(x + 2)$.
Note that if there is a single solution then we have to use it twice.

6. (a) $x^2 + 5x + 4 = (x + 1)(x + 4)$.
   (b) $x^2 - 5x + 6 = (x - 2)(x - 3)$.
   (c) $x^2 - 4x = x(x - 4)$.
   (d) $x^2 - 4 = (x - 2)(x + 2)$.
   (e) $x^2 + x - 12 = (x - 3)(x + 4)$.